INTEGRALES: INDEFINIDAS Y DEFINIDAS

Calcula las siguientes integrales:

INTEGRALES INMEDIATAS

1) $\int\left(3x^3-5x^2+3x+4\right)\,\mathrm{d}x=3\frac{x^4}{4}-5\frac{x^3}{3}+3\frac{x^2 2}{}+4x+C$

2) $\int\left(\operatorname{sen}x+7\cos x-1\right)\,\mathrm{d}x=\int\operatorname{sen}x\,\mathrm{d}x+7\int\cos x\,\mathrm{d}x-\int 1\,\mathrm{d}x=-\cos x+7\operatorname{sen}x-x+C$

3) $\int\operatorname{tg}^2 x\,\mathrm{d}x=\operatorname{tg}x+x+C$

4) $\int\left(\sqrt{x}-2\right)\,\mathrm{d}x=\int\sqrt{x}\,\mathrm{d}x-2\int 1\,\mathrm{d}x=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}-2x=\frac{2}{3}x\sqrt{x}-2x+C$

5) $\int\frac{2}{\sqrt{x}}\,\mathrm{d}x=2\int x^{-1/2}\,\mathrm{d}x=2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}=4\sqrt{x}+C$

6) $\int\frac{x^3-2x^2+4x}{x}\,\mathrm{d}x=\int\frac{x^3}{x}\,\mathrm{d}x-2\int\frac{x^2}{x}\,\mathrm{d}x+4\int\frac{x}{x}\,\mathrm{d}x=\frac{x^3}{3}-x^2+4x+C$

7) $\int\left(4x+3\right)^2\,\mathrm{d}x=\frac{1}{4}\int 4\left(4x+3\right)^2\,\mathrm{d}x=\frac{1}{4}\frac{\left(4x+3\right)^3}{3}=\frac{1}{12}\left(4x+3\right)^3+C$

8) $\int\frac{\left(2x-1\right)^2}{2x}\,\mathrm{d}x=\int\frac{2x^2-4x+1}{2x}\,\mathrm{d}x=\int\frac{2x^2}{2x}\,\mathrm{d}x-\int\frac{4x}{2x}\,\mathrm{d}x+\frac{1}{2}\int\frac{1}{x}\,\mathrm{d}x=x^2-2x+\frac{1}{2}\ln|x|+C$

9) $\int\left(2\sqrt{x}-\sqrt[3]{x}-x^4\right)\,\mathrm{d}x=\int\left(2x^{1/2}-x^{1/3}-x^4\right)\,\mathrm{d}x=2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}-\frac{x^5}{5}=$

$=\frac{x\sqrt{x}}{3}-\frac{3}{4}x\cdot\sqrt[3]{x}-\frac{1}{5}x^5+C$

10) $\int\left(\frac{3}{x}-\frac{x}{3}\right)\,\mathrm{d}x=3\int\frac{1}{x}\,\mathrm{d}x-\frac{1}{3}\int x\,\mathrm{d}x=3\ln|x|-\frac{1}{3}\frac{x^2}{2}=3\ln|x|-\frac{1}{6}x^2+C$

11) $\int\frac{2e^x+e^{2x}}{e^x}\,\mathrm{d}x=2\int\frac{e^x}{e^x}\,\mathrm{d}x+\int\frac{e^{2x}}{e^x}\,\mathrm{d}x=2x+\int e^x\,\mathrm{d}x=2x+e^x+C$

12) $\int\frac{2}{1+x^2}\,\mathrm{d}x=2\int\frac{1}{1+x^2}\,\mathrm{d}x=2\operatorname{arctg}x+C$

13) $\int\left(4x+2\right)\left(x-1\right)\,\mathrm{d}x=\int\left(4x^2-2x-2\right)\,\mathrm{d}x=\frac{4}{3}x^3-x^2-2x+C$

14) $\int 5^x \ \mathrm{d}x = \dfrac{5^x}{\ln 5} + C$

15) $\int e^{2x+1} \ \mathrm{d}x = \dfrac{1}{2}\int 2e^{2x+1} \ \mathrm{d}x = \dfrac{1}{2}e^{2x+1} + C$

16) $\int \cos(2x+5) \ \mathrm{d}x = \dfrac{1}{2}\int 2\cos(2x+5) \ \mathrm{d}x = \dfrac{1}{2}\mathrm{sen}(2x+5) + C$

17) $\int \mathrm{tg}\, x \ \mathrm{d}x = \int \dfrac{\mathrm{sen}\, x}{\cos x} \ \mathrm{d}x = -\int \dfrac{-\mathrm{sen}\, x}{\cos x} \ \mathrm{d}x = -\ln|\cos x| + C$

18)

$$\int \sqrt{1-4x} \ \mathrm{d}x = \int (1-4x)^{1/2} \ \mathrm{d}x = \begin{bmatrix} t = 1-4x \\ \mathrm{d}t = -4\mathrm{d}x \to \mathrm{d}x = -\dfrac{1}{4}dt \end{bmatrix} = \int t^{1/2}\left(-\frac{1}{4}\right) \mathrm{d}t = -\frac{1}{4}\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} =$$

19) $\int \dfrac{3x-2}{x^2-2x+1} dx$

Descomponemos en fracciones simples:

$x^2 - 2x + 1 = 0 \Rightarrow x = 1$ (doble)

$$\frac{3x-2}{x^2-2x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1)+B \Rightarrow \begin{cases} A = 3 \\ -2 = -A + B \end{cases} \Rightarrow (A,B) = (3,1)$$

$$\frac{3x-2}{x^2-2x+1} = \frac{3}{x-1} + \frac{1}{(x-1)^2}$$

Integramos:

$$\int \frac{3x-2}{x^2-2x+1} dx = 3\int \frac{1}{x-1} dx + \int \frac{1}{(x-1)^2} dx = 3\log|x-1| + \int (x-1)^{-2} dx = 3\log|x-1| - \frac{1}{x-1} + C$$

donde $\log = \ln$ es el logaritmo natural o neperiano.

20) $\int \dfrac{1}{\sqrt{x}(x+1)} dx$

$$\int \frac{1}{\sqrt{x}(x+1)} dx = \begin{bmatrix} t = \sqrt{x} \xrightarrow{\text{derivando}} dt = \dfrac{1}{2\sqrt{x}} dx \\ t^2 = x \end{bmatrix} = 2\int \frac{1}{1+t^2} dt = 2\,\mathrm{arctg}\, t = 2\,\mathrm{arctg}\left(\sqrt{x}\right) + C$$

21) $\int \dfrac{x^3 + 2x^2 + x - 10}{x^2 + x - 2} dx$

Efectuamos la división:

$$\begin{array}{rl} x^3+2x^2+x-10 & \underline{|x^2+x-2} \\ \underline{-x^3-x^2+2x} & x+1 \\ x^2+3x-10 & \\ \underline{-x^2-x+2} & \\ 2x-8 & \end{array} \qquad \frac{x^3+2x^2+x-10}{x^2+x-2}=x+1+\frac{2x-8}{x^2+x-2}$$

Descomponemos en fracciones simples:

$x^2+x-2=(x-1)(x+2)$

$$\frac{2x-8}{x^2+x-2}=\frac{A}{x-1}+\frac{B}{x+2}=\frac{A(x+2)+B(x-1)}{(x-1)(x+2)}\Rightarrow$$

$$\Rightarrow 2x-8=A(x+2)+B(x-1)\Rightarrow\begin{cases}\text{Si } x=1\Rightarrow -6=3A\Rightarrow A=-2\\ \text{Si } x=-2\Rightarrow -12=-3B\Rightarrow B=4\end{cases}$$

$$\frac{2x-8}{x^2+x-2}=\frac{-2}{x-1}+\frac{4}{x+2}$$

Integramos:

$$\int\frac{x^3+2x^2+x-10}{x^2+x-2}dx=\int(x+1)dx+\int\frac{-2}{x-1}dx+\int\frac{4}{x+2}=$$

$$=\frac{x^2}{2}+x-2\log|x-1|+4\log|x+2|+C$$

donde $\log=\ln$ es el logaritmo natural o neperiano.

22) $\int x^2\log x dx$

$$\int x^2\log xdx=\begin{bmatrix}u=\log x\xrightarrow{\text{derivamos}}du=\frac{1}{x}dx\\ dv=x^2dx\xrightarrow{\text{integramos}}v=\frac{x^3}{3}\end{bmatrix}=\frac{1}{3}x^3\log x-\int\frac{x^3}{3}\frac{1}{x}dx=\frac{1}{3}x^3\log x-\frac{1}{3}\int x^2dx=$$

$$=\frac{1}{3}x^3\log x-\frac{1}{3}\frac{x^3}{3}=\frac{1}{3}x^3\log x-\frac{x^3}{9}+C$$

donde $\log=\ln$ es el logaritmo natural o neperiano.

23) $\int\frac{2\operatorname{sen}x\cos x}{1+\operatorname{sen}^2 x}dx$

$$\int\frac{2\operatorname{sen}x\cos x}{1+\operatorname{sen}^2 x}dx=\begin{bmatrix}t=1+\operatorname{sen}^2 x\\ dt=2\operatorname{sen}x\cos xdx\end{bmatrix}=\int\frac{1}{t}dt=\log|t|=\log\left|1+\operatorname{sen}^2 x+C\right|$$

donde $\log=\ln$ es el logaritmo natural o neperiano.

24) $\int\frac{x^2+x-4}{x^3-4x}dx$

Descomponemos en fracciones simples:

$x^3-4x=x\left(x^2-4\right)=x\left(x+2\right)\left(x-2\right)$

$$\frac{x^2+x-4}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}=\frac{A\left(x-2\right)\left(x+2\right)+Bx\left(x+2\right)+Cx\left(x-2\right)}{x\left(x-2\right)\left(x+2\right)}\Rightarrow$$

$$\Rightarrow A\left(x-2\right)\left(x+2\right)+Bx\left(x+2\right)+Cx\left(x-2\right)=x^2+x-4\Rightarrow$$

$$\Rightarrow\begin{cases}x=2\Rightarrow A\left(2-2\right)\left(2+2\right)+2B\left(2+2\right)+2C\left(2-2\right)=2^2+2-4\Rightarrow B=\dfrac{1}{4}\\ x=0\Rightarrow A\left(0-2\right)\left(0+2\right)+B\cdot 0\cdot\left(0+2\right)+C\cdot 0\cdot\left(0-2\right)=0^2+0-4\Rightarrow A=1\\ x=-2\Rightarrow C=-\dfrac{1}{4}\end{cases}$$

$$\frac{x^2+x-4}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}=\frac{1}{x}+\frac{\frac{1}{4}}{x-2}+\frac{-\frac{1}{4}}{x+2}$$

Integramos:

$$\int\frac{x^2+x-4}{x^3-4x}dx=\int\frac{1}{x}dx+\frac{1}{4}\int\frac{1}{x-2}dx-\frac{1}{4}\int\frac{1}{x+2}dx=\log\left|x\right|+\frac{1}{4}\log\left|x-2\right|-\frac{1}{4}\log\left|x+2\right|+C$$

donde $\log=\ln$ es el logaritmo natural o neperiano.

25) $\displaystyle\int\frac{2\cos x}{1+\operatorname{sen}^2 x}dx$

$$\int\frac{2\cos x}{1+\operatorname{sen}^2 x}dx=\begin{bmatrix}t=\operatorname{sen}x\\ dt=\cos x\,dx\end{bmatrix}=2\int\frac{1}{1+t^2}dt=2\operatorname{arctg}t=2\operatorname{arctg}\left(\operatorname{sen}x\right)+C$$

26) $\displaystyle\int e^{x+e^x}dx$

$$\int e^{x+e^x}dx=\int e^x e^{e^x}dx=\begin{bmatrix}t=e^x\\ dt=e^x dx\end{bmatrix}=\int e^t dt=e^t=e^{e^x}+C$$

27) $\displaystyle\int\left(x^2+2x\right)\log x\,dx$, donde $\log$ representa el logaritmo natural.

$$\int\left(x^2+2x\right)\log x\,dx=\begin{bmatrix}u=\log x\xrightarrow{\text{derivamos}}du=\dfrac{1}{x}dx\\ dv=\left(x^2+2x\right)\xrightarrow{\text{integramos}}v=\dfrac{x^3}{3}+x^2\end{bmatrix}=\left(\frac{x^3}{3}+x^2\right)\log x-\int\left(\frac{x^3}{3}+x^2\right)\frac{1}{x}dx=$$

$$=\left(\frac{x^3}{3}+x^2\right)\log x-\int\left(\frac{x^2}{3}+x\right)dx=\left(\frac{x^3}{3}+x^2\right)\log x-\frac{x^3}{9}-\frac{x^2}{2}+C$$

donde $\log=\ln$ es el logaritmo natural o neperiano.

28) $\displaystyle\int\operatorname{sen}^2 x\cos x\,dx$

Es inmediata de tipo potencial: $\int \operatorname{sen}^2 x \cos x dx = \frac{\operatorname{sen}^3 x}{3} + C$

También se puede calcular mediante un cambio de variable:

$$\int \operatorname{sen}^2 x \cos x dx = \begin{bmatrix} t = \operatorname{sen} x \\ dt = \cos x dx \end{bmatrix} = \int t^2 dt = \frac{t^3}{3} = \frac{\operatorname{sen}^3 x}{3} + C$$

29) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \begin{bmatrix} t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}} dx \end{bmatrix} = \int 2e^t dt = 2e^t = 2e^{\sqrt{x}} + C$$

30) $\int \frac{x^2 - 3x + 1}{x^3 - 5x^2 + 8x - 4} dx$

Descomponemos en fracciones simples:

$x^3 - 5x^2 + 8x - 4 = (x-1)(x-2)^2$ (aplicando la regla de Ruffini)

$$\frac{x^2 - 3x + 1}{x^3 - 5x^2 + 8x - 4} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{A(x-2)^2 + B(x-1)(x-2) + C(x-1)}{(x-1)(x-2)^2} \Rightarrow$$

$$\Rightarrow x^2 - 3x + 1 = A(x-2)^2 + B(x-1)(x-2) + C(x-1) \Rightarrow$$

$$\Rightarrow \begin{cases} \text{Si } x = 1 \Rightarrow A = -1 \\ \text{Si } x = 2 \Rightarrow C = -1 \\ \text{Derivamos dos veces: } x^2 - 3x + 1 = -(x-2)^2 + B(x-1)(x-2) - (x-1) \Rightarrow \\ 2x - 3 = -2(x-2) + 2Bx - 1 \\ 2 = -2 + 2B \Rightarrow B = 2 \end{cases}$$

$$\Rightarrow \frac{x^2 - 3x + 1}{x^3 - 5x^2 + 8x - 4} = \frac{-1}{x-1} + \frac{2}{x-2} + \frac{-1}{(x-2)^2}$$

Integramos:

$$\int \frac{x^2 - 3x + 1}{x^3 - 5x^2 + 8x - 4} dx = \int \frac{-1}{x-1} dx + \int \frac{2}{x-2} dx + \int \frac{-1}{(x-2)^2} dx = -\log|x-1| + 2\log|x-2| + \frac{1}{x-2} + C$$

donde $\log = \ln$ es el logaritmo natural o neperiano.

31) $\int x \log x dx$

$$\int x\log x dx=\begin{bmatrix} u=\log x \xrightarrow{\text{derivamos}} du=\frac{1}{x}dx \\ dv=xdx \xrightarrow{\text{integramos}} v=\frac{x^2}{2} \end{bmatrix}=\frac{1}{2}x^2\log x-\int\frac{x^2}{2}\frac{1}{x}dx=\frac{1}{2}x^2\log x-\frac{1}{2}\int xdx=$$

$$=\frac{1}{2}x^2\log x-\frac{1}{2}\frac{x^2}{2}=\frac{1}{2}x^2\log x-\frac{1}{4}x^2+C$$

donde $\log=\ln$ es el logaritmo natural o neperiano.

32) $\int\frac{\sqrt{x}}{1+\sqrt{x}}dx$

$$\int\frac{\sqrt{x}}{1+\sqrt{x}}dx=\begin{bmatrix} t=\sqrt{x}\Rightarrow t^2=x \\ dx=2tdt \end{bmatrix}=\int\frac{t}{1+t}2tdt=2\int\frac{t^2}{1+t}dt$$

Descomponemos en fracciones simples $\frac{t^2}{1+t}$:

$$\begin{array}{r|l} t^2 & t+1 \\ \hline \underline{-t^2-t} & t-1 \\ -t & \\ \underline{t+1} & \\ 1 & \end{array} \quad\Rightarrow \frac{t^2}{t-1}=t-1+\frac{1}{t+1}$$

$$\frac{t^2}{t-1}=t-1+\frac{1}{t+1}$$

Integramos:

$$\int\frac{t^2}{t-1}dt=\int(t-1)dt+\int\frac{1}{t+1}dt=\frac{t^2}{2}-t+\log|t+1|$$

$$\int\frac{\sqrt{x}}{1+\sqrt{x}}dx=\begin{bmatrix} t=\sqrt{x}\Rightarrow t^2=x \\ dx=2tdt \end{bmatrix}=\int\frac{t}{1+t}2tdt=2\int\frac{t^2}{1+t}dt=2\left(\frac{t^2}{2}-t+\log|t+1|\right)=$$

$$=2\left(\frac{x}{2}-\sqrt{x}+\log\left|\sqrt{x}+1\right|\right)+C$$

donde $\log=\ln$ es el logaritmo natural o neperiano.

33) $\int\frac{\cos x}{1+\operatorname{sen}^2 x}dx$

$$\int\frac{\cos x}{1+\operatorname{sen}^2 x}dx=\begin{bmatrix} t=\operatorname{sen} x \\ dt=\cos xdx \end{bmatrix}=\int\frac{1}{1+t^2}dt=\operatorname{arctg} t=\operatorname{arctg}(\operatorname{sen} x)+C$$

También se puede calcular como una integral inmediata:

$$\int \frac{\cos x}{1+\text{sen}^2 x}dx = \int \frac{\overbrace{\cos x}^{\text{derivada}}}{1+\left(\underbrace{\text{sen}\, x}_{\text{función}}\right)^2}dx = \text{arctg}\left(\text{sen}\, x\right)+C$$

34) $\int \frac{x+2}{\sqrt{x+1}}dx$

$$\int \frac{x+2}{\sqrt{x+1}}dx = \begin{bmatrix} t = x+1 \\ dt = dx \end{bmatrix} = \int \frac{t+1}{\sqrt{t}}dt = \int \frac{t}{\sqrt{t}}dt + \int \frac{1}{\sqrt{t}}dt = \int t^{\frac{1}{2}}dt + \int t^{-\frac{1}{2}}dt = \frac{2}{3}\sqrt{t^3} + 2\sqrt{t} =$$

$$= \frac{2}{3}\sqrt{\left(x+1\right)^3} + 2\sqrt{x+1} = \frac{2\left(x+1\right)\sqrt{x+1}}{3} + 2\sqrt{x+1} = \frac{\left(2x+4\right)\sqrt{x+1}}{3} + C$$

Con otro cambio de variable:

$$\int \frac{x+2}{\sqrt{x+1}}dx = \begin{bmatrix} t^2 = x+1 \\ 2tdt = dx \end{bmatrix} = \int \frac{t^2+1}{t}2tdt = 2\int \left(t^2+1\right)dt = 2t\left(\frac{t^3}{3}+t\right) = 2\left(\frac{\sqrt{\left(x+1\right)^3}}{3} + \sqrt{x+1}\right) =$$

$$= \frac{\left(2x+4\right)\sqrt{x+1}}{3} + C$$

35) $\int \frac{2x^3-9x^2+9x+6}{x^2-5x+6}dx$

Descomponemos en fracciones simples

$$\begin{array}{rl} 2x^3-9x^2+9x+6 & \lfloor x^2-5x+6 \\ \underline{-2x^3+10x^2-12x} & 2x+1 \\ x^2-3x+6 & \\ \underline{-x^2+5x-6} & \\ 2x & \end{array} \Rightarrow \frac{2x^3-9x^2+9x+6}{x^2-5x+6} = 2x+1+\frac{2x}{x^2-5x+6}$$

$$\frac{2x}{x^2-5x+6} = \frac{2x}{\left(x-2\right)\left(x-3\right)} = \frac{A}{x-2}+\frac{B}{x-3} = \frac{A\left(x-3\right)+B\left(x-2\right)}{\left(x-2\right)\left(x-3\right)} \Rightarrow$$

$$\Rightarrow 2x = A\left(x-3\right)+B\left(x-2\right) \Rightarrow \begin{cases} x=2 \Rightarrow A=-4 \\ x=3 \Rightarrow B=6 \end{cases}$$

$$\Rightarrow \frac{2x}{x^2-5x+6} = \frac{-4}{x-2}+\frac{6}{x-3}$$

Integramos

$$\int \frac{2x^3-9x^2+9x+6}{x^2-5x+6}dx = \int \left(2x+1\right)dx + \int \frac{2x}{x^2-5x+6}dx = \int 2xdx + \int 1dx + \int \frac{-4}{x-2}dx + \int \frac{6}{x-3}dx =$$

$$= x^2+x-4\log\left|x-2\right|+6\log\left|x-3\right|+C$$

donde $\log = \ln$ es el logaritmo natural o neperiano.

36) $\int \sqrt{1-x^2}\,dx = \begin{bmatrix} x=\operatorname{sen} t \\ dx = \cos t dt \end{bmatrix} = \int \sqrt{1-\operatorname{sen}^2 t}\cos t dt \underset{(1)}{=} \int \sqrt{\cos^2 t}\cos t dt = \int \cos^2 t dt =$

$$= \int \frac{1+\cos(2t)}{2}dt = \frac{1}{2}\left(t + \int \cos(2t)\,dt\right) = \frac{1}{2}\left(t + \frac{\operatorname{sen}(2t)}{2}\right) = \frac{1}{2}\operatorname{arcsen} x + \frac{1}{4}\operatorname{sen}(2\operatorname{arcsen} x) \underset{(2)}{=}$$

$$= \frac{1}{2}\operatorname{arcsen} x + \frac{1}{4}2\operatorname{sen}(\operatorname{arcsen} x)\cos(\operatorname{arcsen} x) \underset{(3)}{=} \frac{1}{2}\operatorname{arcsen} x + \frac{1}{2}x\sqrt{1-x^2} + C$$

Donde en (1) hemos tenido en cuenta que $\operatorname{sen}^2 x + \cos^2 x = 1 \Rightarrow 1 - \operatorname{sen}^2 x = \cos^2 x$, en (2) la fórmula del seno del ángulo doble, $\operatorname{sen}(2x) = 2\operatorname{sen} x \cos x$, y en (3):

$$\operatorname{sen}^2 x + \cos^2 x = 1 \Rightarrow \cos x = \sqrt{1-\operatorname{sen}^2 x} \Rightarrow \cos(\operatorname{arcsen} x) = \sqrt{1-x^2}$$

37) $\int \frac{e^{2x}+e^x}{1+e^{2x}}dx$

$$\int \frac{e^{2x}+e^x}{1+e^{2x}}dx = \int \frac{(e^x+1)e^x}{1+e^{2x}}dx = \begin{bmatrix} t = e^x \\ dt = e^x dx \end{bmatrix} = \int \frac{t+1}{1+t^2}dt = \int \frac{t}{1+t^2}dt + \int \frac{1}{1+t^2}dx =$$

$$= \frac{1}{2}\log\left|1+t^2\right| + \operatorname{arctg} t = \frac{1}{2}\log\left|1+e^{2x}\right| + \operatorname{arctg}(e^x) + C$$

donde $\log = \ln$ es el logaritmo natural o neperiano.

38) $\int \operatorname{tg} x dx$

$$\int \operatorname{tg} x dx = \int \frac{\operatorname{sen} x}{\cos x}dx = -\log\left|\cos x\right| + C \quad \text{(inmediata)}$$

También se puede calcular por cambio de variable:

$$\int \operatorname{tg} x dx = \int \frac{\operatorname{sen} x}{\cos x}dx = \begin{bmatrix} t = \cos x \\ dt = -\operatorname{sen} x dx \end{bmatrix} = -\int \frac{1}{t}dt = -\log|t| = -\log\left|\cos x\right| + C$$

39) $\int \frac{2}{1+\sqrt{x}}dx$

$$\int \frac{2}{1+\sqrt{x}}dx = \begin{bmatrix} t^2 = x \\ 2tdt = dx \end{bmatrix} = \int \frac{2}{1+t}2tdt = 4\int \frac{t}{1+t}dt$$

Descomponemos en fracciones simples $\frac{t}{1+t}$:

$$\begin{array}{r|l} t & t+1 \\ \hline -t+1 & 1 \\ \hline -1 & \end{array} \quad \Rightarrow \quad \frac{t}{1+t} = 1 - \frac{1}{1+t}$$

Integramos

$$\int \frac{t}{1+t}dt = \int 1dt - \int \frac{1}{1+t}dt = t - \log|1+t|$$

Así:

$$\int \frac{2}{1+\sqrt{x}}dx = \begin{bmatrix} t^2 = x \\ 2tdt = dx \end{bmatrix} = \int \frac{2}{1+t}2tdt = 4\int \frac{t}{1+t}dt = 4\left(t - \log|1+t|\right) = 4\left(\sqrt{x} - \log\left|1+\sqrt{x}\right|\right) + C$$

40) $\int \frac{x+2}{x^2-2x+1}dx$

Descomponemos en fracciones simples:

$$\frac{x+2}{x^2-2x+1} = \frac{x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} \Rightarrow x+2 = A(x-1)+B$$

$$\Rightarrow \begin{cases} x = 1 \Rightarrow B = 3 \\ \text{Derivamos } x+2 = A(x-1)+B \\ \qquad 1 = A \end{cases}$$

$$\Rightarrow \frac{x+2}{x^2-2x+1} = \frac{1}{x-1} + \frac{3}{(x-1)^2}$$

Integramos:

$$\int \frac{x+2}{x^2-2x+1}dx = \int \frac{1}{x-1}dx + \int \frac{3}{(x-1)^2}dx = \log|x-1| - \frac{3}{x-1} + C$$

41) $\int \left[\cos(2x) + \operatorname{sen} x \cos x\right]dx$

$$\int \left[\cos(2x) + \operatorname{sen} x \cos x\right]dx = \int \cos(2x)\,dx + \int \operatorname{sen} x \cos x dx = \frac{1}{2}\operatorname{sen}(2x) + \frac{\operatorname{sen}^2 x}{2} + C$$

42) $\int \frac{x^3-1}{x+2}dx$

Descomponemos en fracciones simples:

$$\begin{array}{llll} x^3 & & -1 & \underline{|x+2\quad} \\ \underline{-x^3-2x^2} & & & x^2-2x+4 \\ \quad 2x^2+4x-1 & & & \\ \underline{\ -2x^2+4x} & & & \\ \qquad 8x-1 & & & \\ \qquad \underline{-8x-8} & & & \\ \qquad\quad -9 & & & \end{array}$$

$$\Rightarrow \frac{x^3+1}{x+2} = x^2 - 2x + 4 - \frac{9}{x+2}$$

Integramos:

$$\int \frac{x^3+1}{x+2}dx = \int \left(x^2-2x+4\right)dx - 9\int \frac{1}{x+2}dx = \frac{x^3}{3} - x^2 + 4x - 9\log|x+2| + C$$

43) $\displaystyle\int \frac{3x}{x^2+2x+3}dx$

El polinomio x^2+2x+3 no tiene raíces reales. Ahora bien:

$$\int \frac{3x}{x^2+2x+3}dx = \frac{1}{2}\int \frac{2\cdot 3x}{x^2+2x+3}dx = \frac{1}{2}\int \frac{6x}{x^2+2x+3}dx = \frac{1}{2}\int \frac{6x+6-6}{x^2+2x+3}dx =$$

$$= \frac{1}{2}\int \frac{6x+6}{x^2+2x+3}dx + \frac{1}{2}\int \frac{-6}{x^2+2x+3}dx = \underbrace{\frac{1}{2}\int \frac{3\left(\overbrace{2x+2}^{\text{derivada del denominador}}\right)}{x^2+2x+3}dx}_{\text{inmediata}} + \frac{1}{2}\int \frac{-6}{x^2+2x+3}dx$$

Calculamos $\displaystyle\int \frac{-6}{x^2+2x+3}dx$:

$$\frac{1}{2}\int \frac{-6}{x^2+2x+3}dx \underset{(1)}{=} -3\int \frac{1}{(x+1)^2+2}dx = -3\int \frac{\frac{1}{2}}{\frac{(x+1)^2+2}{2}}dx = -\frac{3}{2}\int \frac{1}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}dx =$$

$$= -\frac{3}{2}\sqrt{2}\int \frac{\frac{1}{\sqrt{2}}}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}dx = -\frac{3\sqrt{2}}{2}\operatorname{arctg}\left(\frac{x+1}{\sqrt{2}}\right)$$

donde en (1) hemos tenido en cuenta que $(x+1)^2+2 = x^2+1+2x+2 = x^2+2x+3$

Así:

$$\int \frac{3x}{x^2+2x+3}dx = \frac{3}{2}\log\left|x^2+2x+3\right| - \frac{3\sqrt{2}}{2}\operatorname{arctg}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

44) $\displaystyle\int_0^1 (x+1)e^{-x}dx$

Calculamos una primitiva de $(x+1)e^{-x}$:

$$\int (x+1)e^{-x}dx = \left[\begin{array}{l} u = x+1 \xrightarrow{\text{derivamos}} du = dx \\ dv = e^{-x}dx \xrightarrow{\text{integramos}} v = -e^{-x} \end{array}\right] = -(x+1)e^{-x} + \int e^{-x}dx = -(x+1)e^{-x} - e^{-x} =$$

$$= (-x-2)e^{-x} + C = G(x)$$

Aplicamos la regla de Barrow:

$$\int_0^1 (x+1)e^{-x}dx = G(1) - G(0) = -3e^{-1} - (-2) = 2 - \frac{3}{e}$$

45) $\displaystyle\int_0^{\pi} e^x \operatorname{sen} x dx$

$$\int e^x \operatorname{sen} x dx = \begin{bmatrix} u = e^x \xrightarrow{\text{derivamos}} du = e^x dx \\ dv = \operatorname{sen} x dx \xrightarrow{\text{integramos}} v = -\cos x \end{bmatrix} = -e^x \cos x + \int e^x \cos x dx$$

Calculamos $\int e^x \cos x dx$ por partes:

$\int e^x \cos x dx = \begin{bmatrix} u = e^x \xrightarrow{\text{derivamos}} du = e^x dx \\ dv = \cos x dx \xrightarrow{\text{integramos}} v = \operatorname{sen} x \end{bmatrix} = e^x \operatorname{sen} x - \int e^x \operatorname{sen} x dx$ (donde la última integral es la misma que queremos calcular).

Así:

$$\int e^x \operatorname{sen} x dx = \begin{bmatrix} u = e^x \xrightarrow{\text{derivamos}} du = e^x dx \\ dv = \operatorname{sen} x dx \xrightarrow{\text{integramos}} v = -\cos x \end{bmatrix} = -e^x \cos x + \int e^x \cos x dx =$$

$$= -e^x \cos x + e^x \operatorname{sen} x - \int e^x \operatorname{sen} x dx$$

Si llamamos $I = \int e^x \operatorname{sen} x dx$, se tiene que

$$I = -e^x \cos x + e^x \operatorname{sen} x - I \Rightarrow 2I = -e^x \cos x + e^x \operatorname{sen} x - e^x \cos x + e^x \operatorname{sen} x \Rightarrow$$

$$\Rightarrow I = \frac{-e^x \cos x + e^x \operatorname{sen} x}{2} = \frac{e^x (\operatorname{sen} x - \cos x)}{2}$$

y, por tanto, $\int e^x \operatorname{sen} x dx = \dfrac{e^x (\operatorname{sen} x - \cos x)}{2} + C$

Llamamos $G(x) = \dfrac{e^x (\operatorname{sen} x - \cos x)}{2} + C$ y aplicamos la regla de Barrow:

$$\int_0^\pi e^x \operatorname{sen} x dx = G(\pi) - G(0) = \frac{e^\pi + 1}{2}$$

46) $\int_0^1 (x^2 + x) e^x dx$

$$\int (x^2 + x) e^x dx = \begin{bmatrix} u = x^2 + x \xrightarrow{\text{derivamos}} du = (2x+1) dx \\ dv = e^x dx \xrightarrow{\text{integramos}} v = e^x \end{bmatrix} = (x^2 + x) e^x - \int (2x+1) e^x dx =$$

$$= (2x+1) e^x - \left[(2x+1) e^x - 2e^x \right] + C = G(x)$$

ya que

$$\int (2x+1) e^x dx = \begin{bmatrix} u = 2x+1 \xrightarrow{\text{derivamos}} du = 2dx \\ dv = e^x dx \xrightarrow{\text{integramos}} v = e^x \end{bmatrix} = (2x+1) e^x - 2\int e^x dx = (2x+1) e^x - 2e^x$$

Aplicando la regla de Barrow:

$$\int_0^1 (x^2 + x) e^x dx = G(1) - G(0) = e - 1$$

47) $\int_e^{e^3} \dfrac{\log x}{x} dx$

Calculamos una primitiva:

$$\int \frac{\log x}{x}dx = \begin{bmatrix} t = \log x \\ dt = \frac{1}{x}dx \end{bmatrix} = \int t dt = \frac{t^2}{2} = \frac{1}{2}\log^2 x + C = G(x)$$

(dicha primitiva, de hecho, es inmediata: $\int \frac{\log x}{x}dx = \int \underbrace{\frac{1}{x}}_{\text{derivada}} \underbrace{\log x}_{\text{función}} dx = \frac{\log^2 x}{2} + C$)

Aplicamos la regla de Barrow:

$$\int_e^{e^3} \frac{\log x}{x}dx = G(e^3) - G(e) = 4$$

48) $\int_0^1 \frac{2\,\text{arctg}\, x}{1+x^2}dx$

Calculamos una primitiva:

$$\int \frac{2\,\text{arctg}\, x}{1+x^2}dx = 2\int \frac{\text{arctg}\, x}{1+x^2}dx = 2\int \underbrace{\frac{1}{1+x^2}}_{\text{derivada}} \underbrace{\text{arctg}\, x}_{\text{función}} dx = 2\frac{\text{arctg}^2\, x}{2} = \text{arctg}^2\, x + C = G(x)$$

Aplicamos la regla de Barrow:

$$\int_0^1 \frac{2\,\text{arctg}\, x}{1+x^1}dx = G(1) - G(0) = \left(\frac{\pi}{4}\right)^2 - 0^2 = \frac{\pi^2}{16}$$

Directamente, por cambio de variable:

$$\int_0^1 \frac{2\,\text{arctg}\, x}{1+x^2}dx = \begin{bmatrix} t = \text{arctg}\, x \rightarrow \begin{cases} x = 1 \Rightarrow t = \frac{\pi}{4} \\ x = 0 \Rightarrow t = 0 \end{cases} \\ dt = \frac{1}{1+x^2}dx \end{bmatrix} = 2\int_0^{\frac{\pi}{4}} t dt = 2\frac{t^2}{2}\Big|_0^{\frac{\pi}{4}} = t^2\Big|_0^{\frac{\pi}{4}} = \text{arctg}^2\, x\Big|_0^1 = \left(\frac{\pi}{4}\right)^2 - 0^2 = \frac{\pi^2}{16}$$